Original Research Article

Random Walks, Electrical Networks and Pólya's Theorem

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ABSTRACT

A fundamental topic in probability theory is random walks on infinite graphs. In particular, classifying them as recurrent and transient is of utmost importance for the understanding of their behavior. These properties reveal deep connections to electrical networks and highlight the structural properties of the underlying space. Approximately, a random walk is recurrent if it cannot escape to infinity; otherwise, it is transient. Pólya's Theorem characterizes recurrence in dimensions $\mathbb{Z}^d$ for $d = 1,2$ and transience for $d \geq 3$. Building on this theorem, the paper examines how the addition or deletion of edges impacts effective resistance to infinity and, consequently, the recurrence-transience classification. Using the analogy between random walks and electrical networks developed in Lyons and Peres, the paper interprets recurrence as infinite effective resistance and transience as finite resistance to infinity. Tools such as the Nash-Williams Criterion, Rayleigh's Monotonicity Principle, and Thomson's Principle are used to analyze resistance under graph modifications. The study confirms Pólya's Theorem and explains the dimensional threshold: in low dimensions, limited connectivity causes resistance to diverge, ensuring recurrence; in higher dimensions, the abundance of disjoint paths support finite-energy flows, leading to transience. The paper also investigates how adding or deleting edges alters resistance profiles. It is shown that while finite changes typically preserve recurrence or transience, systematic or unbounded modifications can switch the behavior entirely. In short, by combining ideas from probability and electrical network theory, the paper provides insight into how random walks behave in different dimensions and how changes like adding or removing edges can influence that behavior.

Keywords: Random Walks; Electrical Networks; Effective Resistance; Pólya's Theorem; Energy; Probability Theory

INTRODUCTION

At its core, this paper focuses on the concept of random walks which is one of the most fundamental stochastic processes in probability theory (1). A random walk on a graph is a sequence of moves where, at each step, a walker located at a vertex moves to one of its neighboring vertices which are chosen with a certain probability (2). If every neighboring vertex is chosen with equal probability, then the random walk is classified as a simple symmetric random walk (3). Unless otherwise stated, the random walks considered are often simple and symmetric. A central question in the study of random walks on infinite graphs is whether the walker is recurrent or transient. A walk is said to be recurrent if it returns to its starting point with probability one. Otherwise, if there is a positive

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Accepted October 29, 2025
https://doi.org/10.70251/HYJR2348.36312323

probability that the walk will escape to infinity and never return, it is called transient.

While random walks originate from probability theory, their behavior can also be studied through the framework of electrical networks, a fundamental concept in physics. Electrical networks are interconnections of active and passive electrical components. These components include batteries, inductances, and conductors. In this paper, random walks and their aspects are proven by using electrical networks. The concepts of electrical networks and random walks craft a certain connection between probability theory and physics. The most well-known foundation for this connection is Pólya's Theorem, formulated in 1921. In "Two Incidents" (2) Pólya depicts how he managed to achieve his theory of random walks in Euclidian lattices (4) by an incident:

> - *". . . he and his fiancée (would) also set out for a stroll in the woods, and then suddenly I met them there. And then I met them the same morning repeatedly, I don't remember how many times, but certainly much too often and I felt embarrassed: It looked as if I was snooping around which was, I assure you, not the case. I met them by accident - but how likely was it that it happened by accident and not on purpose?"*

In the theorem following this incident, Pólya explores random walks on the integer lattice $\mathbb{Z}^d$. Pólya proved that simple random walks are recurrent in one and two dimensions $d = 1,2$ but become transient for all higher dimensions $d \geq 3$ (2). The shift in behavior arises from the increased volume of space in higher dimensions, which allows the walker more escape routes. As a result, this decreases the likelihood of return. Below is the mathematical definition of transience and recurrence with equations:

Let $G = (V, E)$ represent a locally finite, connected, and infinite graph, where G is the graph as a whole, V is the set of all vertices in the graph and E is the set of all edges (1). Furthermore, let $(X_n)_{n\geq 0}$ be a simple random walk on the graph of G, where $X_0 = o \in V$ is the starting vertex or the origin. Let P_o denote the probability measure conditioned on $X_0 = o$, thus the probability to return at time n to the starting point o is defined as (5) :

$$p_o^{(n)} \coloneqq P_0(X_n = o)$$

Then, the expected number of returns to the starting point can be shown as follows:

$$\sum_{n=1}^{\infty} P_0(X_n = o)$$

The random walk starting from o is recurrent if it returns to o with probability 1:

$$P_o(\exists n > 0 : X_n = o) = 1 \leftrightarrow \sum_{n=1}^{\infty} P_0(X_n = o) = \infty$$

However, the random walk starting from o is transient if the return probability is less than 1:

$$P_o(\exists n > 0 : X_n = o) < 1 \leftrightarrow \sum_{n=1}^{\infty} P_0(X_n = o) < \infty$$

These equivalences of left- and right-hand sides in the equation above can be proven using standard arguments.

As a result of his investigations, Pólya was able to provide a rigorous classification for ideal lattices. However, real-world networks and especially electrical networks are rarely so regular. From this conflict raises the central question of this paper: How stable are these recurrence and transience properties under modifications to the graph? More specifically, what implications does the addition or deletion of edges provide regarding these properties? This central question can be extended to showcase whether a recurrent walk remains recurrent or can a transient walk become recurrent under certain perturbations.

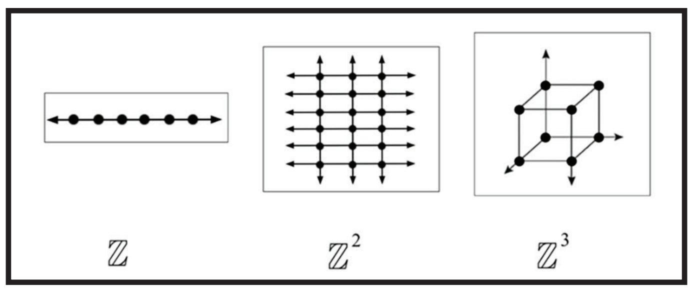

Figure 1. Integer Lattices $\mathbb{Z}^d$ for d=1,2,3 (1) Visualization of lattice structures for random walks illustrating changes with increasing dimensionality. As dimensionality increases, the number of available paths expands, marking the transition from recurrence in lower dimensions (d=1,2) to transience in higher ones (d=3).

This paper aims to answer these questions by exploring the impact of structural changes on the recurrence or transience of random walks. In doing so, it draws on results from modern probability theory, particularly those in Lyons and Peres (1), and incorporates insights from earlier foundational work by Doyle and Snell (6). Furthermore, the objective of the study is to interpret the recurrence or transience behavior of random walks on dimensions of $\mathbb{Z}^d$ for various values of d, focusing on electrical networks and Pólya's Theorem. The paper also analyzes how the addition or deletion of edges affect the long-term return probability of the walk. Lastly, an understanding of the structural thresholds where changes alter the qualitative nature of the walks is established.

The scope of this research is mathematical and theoretical. It focuses exclusively on simple symmetric random walks on infinite graphs and does not use empirical data or simulations. This investigation is framed within the classical probabilistic framework, drawing upon tools such as Markov chains (3), return probability formulas, and foundational theorems, including Thomson's Principle.

In short, by identifying the effects of graph modifications on long-term stochastic behavior, this paper aims both to provide a deeper understanding of dimensional thresholds in random walks and to elucidate insights about how structural changes influence recurrence and transience, enriching the broader theory of stochastic processes on infinite graphs. To position this analysis within existing research, the following section outlines the conceptual framework and the existing literature that links random walks with electrical network theory.

LITERATURE REVIEW

This section focuses on the necessary theoretical background and information required for the reinterpretation of random walks. The first step in this reinterpretation is to reflect graphs as electrical networks and the next steps involve the introduction of key laws utilized for the reformulation.

Graphs as Electrical Networks

In order to reformulate random walks through electrical networks, a connection must be established between the two such that the graph of $G = (V, E)$ must be interpreted as an electrical network (1). In this interpretation:

- G denotes an electrical network instead of a graph.
- V represents junctions or nodes in an electrical circuit instead of vertices.
- E represents wires or resistors that connect these nodes instead of edges.

In this electrical network, each edge $(x, y) \in E$ is assigned to a conductance $c_{xy} > 0$. This conductance's reciprocal is equal to its resistance:

$$r_{xy} = \frac{1}{c_{xy}}$$

As simple random walks are discussed, a unit conductance is assumed on all edges. So:

$$c_{xy} = 1 \text{ and } r_{xy} = 1$$

It allows every edge to behave like a unit resistor and this construction transforms a typical graph into an electrical network where current can flow between nodes. This current is governed by the same laws that are used for electrical circuits. These laws -Ohm's Law, Series and Parallel Laws, Kirchhoff's Laws- will be explained in the following sub-sections.

Ohm's Law

Ohm's law, establishing a proportional relation between current (I), voltage (V), and resistance (R) in the form of $V = I \times R$, is applied to graph structures in this paper and it is reformulated as:

$$I_{xy} = c_{xy} \,(\mathrm{V}(x) - \mathrm{V}(y))$$

Within the context of random walks, the voltage V(x) represents the expected number of visits and the current function represents expected number of times the edge is crossed where a greater number of crossings from x to y corresponds to a higher level of electrical current (1).

Series Law

In an electrical network, if two resistors R_1 and R_2 are connected sequentially such that the only possible way current can flow is through their joined path, then these resistors are connected in series and the total resistance is given by the sum of individual resistances: $R_1 + R_2 = R_{total}$. Within the realm of effective resistance to infinity, Series Law applies when the resistance along outward paths in a network is bounded (1). If there are infinitely many resisters in an electrical network, then total resistance is depicted as:

$$R_{total} = \sum_{n=1}^{\infty} r_n$$

Where, r_n is the minimal resistance of each of the resistors in the network.

Parallel Law

An alternative method of connecting conductors is to join them in parallel in which case they can be treated as a single conductor whose total conductance equals the sum of the individual ones ($c_1 + c_2 = c_{total}$) causing all voltages and currents remain unchanged.

Kirchhoff's Laws

Kirchhoff's Current Law states that at any vertex (except sources/sinks) in a network, the total current entering equals the total current leaving, reflecting a conservation of flow such that no current is lost or gained at intermediate nodes. Kirchhoff's Voltage Law asserts that the sum of voltage differences around any closed loop is zero. This follows from the fact that voltage is a scalar potential making the total net change in voltage around a loop zero.

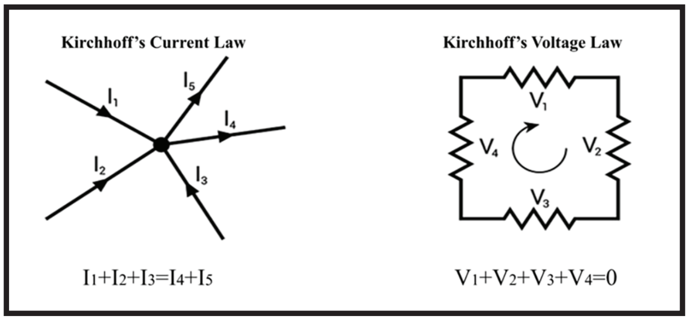

Figure 2. Illustration of Kirchhoff's Laws (7). The left system shows Kirchhoff's Current Law, where total incoming and outgoing currents are equal. The right system illustrates Kirchhoff's Voltage Law, where the sum of potential differences around a loop is zero.

After the presentation of the theoretical background within the existing literature, the following section details the methodology and mathematical reasoning employed in this study.

METHODS AND MATERIALS

The study adopts a deductive and analytical approach, relying on logical reasoning and derivation through formal proofs. All results were obtained through symbolic derivations, theorem-based validation, and logical verification rather than empirical experimentation. The approach puts emphasis on reproducibility by clearly presenting the analytical transformations used to connect probabilistic and physical interpretations of random walks. The analysis proceeds through several stages of reasoning. First, the fundamental concepts of random walks on graphs were formally defined using Markov chain theory. Building on these definitions, the duality between random walks and electrical networks was established by deriving relationships between transition probabilities, conductances, and potential differences. By allowing random walk behavior to be expressed through voltage, current, and resistance relations, this step enabled the use of physical laws in a mathematical context. Next, using Thomson's Principle and Rayleigh's Monotonicity Law, formulas for the effective resistance of infinite lattices under various dimensional conditions were derived. The mathematical reasoning for the derivations was symbolic rather than computational, relying on algebraic manipulation of infinite series, boundary-limit and energy minimization arguments. These methods were subsequently applied to analyze how local modifications and especially the addition or removal of edges affect general network properties. All derivations were verified for internal consistency against results obtained from independent formulations of effective resistance, ensuring mathematical rigor at every step. These logically progressive steps behind the analysis enable the methodology to produce clearly verifiable routes from conceptual definitions to derived results.

Having established methodology and mathematical consistency of the study, the next section presents the core concepts of electrical network theory that are fundamental for capturing random walks on graphs.

Effective Resistance

Exploring effective resistance starts with letting $a, b \in V$ be two vertices in a connected graph G. Effective resistance is denoted by $R_{eff}(a \leftrightarrow b)$ and defined as the potential difference required to drive one unit of current from a to b (6). In other words, effective resistance is the total energy needed to sustain a flow that is minimized over all possible unit flows and it can be shown as follows:

$$R_{eff}(a \leftrightarrow b) = \inf_{\theta \in F_{a \to b}} \varepsilon(\theta)$$

Where,

- $F_{a \to b}$ represents all unit flows between a and b.
- $\varepsilon(\theta)$ represents the energy of the flow.

One of the most important features of this resistance is that it adheres to the properties of physical resistance as it is symmetrical, so:

$$R(a \leftrightarrow b) = R(b \leftrightarrow a)$$

On the other hand, in infinite graphs, the resistance from a vertex to infinity is considered and denoted as: R_{eff} (a ↔ ∞). This type of resistance is defined by using exhaustion limits:

$$R_{eff}(a \leftrightarrow \infty) = \lim_{n \to \infty} R(a \leftrightarrow z_n)$$

Where,

- z_n is the boundary vertex that is formed when all vertices outside a large finite subgraph containing *a* is identified.

Effective Conductance

Effective conductance, in the simplest terms, is the reciprocal of effective resistance:

$$C_{eff}(a \leftrightarrow b) = \frac{1}{R_{eff}(a \leftrightarrow b)}$$

It can be inferred that effective conductance reflects the ease with which current flows from point *a* to point *b*. In the following sections, the positive/negative or zero values of conductance to infinity will be observed.

Energy

Energy, the ability to do work, is the fundamental source of life, production and physics. This concept is so important that the U.S. Energy Information Administration (8) explains it as follows:

> "Moder*n civilization is possible because people have learned how to change energy from one form to another and then use* it to do work."

An antisymmetric function representing a flow of current through the edges of a graph *G* = (*V, E*) is formed to study energy such that:

$$\theta : \mathrm{E} \to \mathbb{R}$$

Then the formula of physical dissipation of power (P) in an electrical resistor is employed to illustrate energy as a function:

$$P = I^2R = \theta^2 r$$

To incorporate random walks to the formula above, the energy $\varepsilon(\theta)$ that is associated with flow θ is represented as follows (1):

$$\varepsilon(\theta) := \frac{1}{2}\sum_{(x,y)\in E} r(x,y) \times \theta(x,y)^2 = \frac{1}{2}\sum_{(x,y)\in E} \frac{\theta(x,y)^2}{\mathrm{c}(x,y)}$$

Where,

- the factor of $\frac{1}{2}$ is used to avoid double counting the edges in the graph.

From this equation, it follows that the total energy is the sum of the power dissipated in all resistors or edges. This energy function is derived from the combination of Ohm's Law and the Dirichlet Principle. As a result, since:

$$\theta_f(x,y) := c(x,y) \times (f(x) - f(y))$$

This equation can be plugged into the energy formula:

$$\varepsilon(f) := \frac{1}{2}\sum_{(x,y)\in E} c(x,y) \times (f(x) - f(y))^2$$

The functional above, noted as Dirichlet Energy (6), is associated with the function *f.* If this functional is minimized over all functions *f* subject to boundary conditions, then it provides the harmonic function with the given boundary values. The most important aspect of this result is that the gradient of this harmonic function yields the minimum-energy unit flow. Building upon these core concepts of electrical network theory, the following section introduces the governing principles of the study.

Thomson's Principle

In the simplest form, Thomson's Principle states that among all unit flows from one set to the other, the actual electrical current, which satisfies the properties that govern electrical circuits, is the one that minimizes energy (1). Equivalently, the current represents the unique unit flow with the least energy while all the other unit flows either have greater energy or equal the current.

Thomson's Principle's formalization starts with a finite network *G* = (*V, E*) containing positive edge conductances $c(e) > 0$ and thus resistances. If two disjoint sets are identified – source set A and sink set Z – then a flow θ becomes a unit flow from A to Z if its divergence satisfies $d^*\theta(x) = 0$ for $x \notin \mathrm{A} \cup \mathrm{Z}$ and the total net outflow from A equals 1. The energy of θ is then symbolized as :

$$\varepsilon(\theta) = \frac{1}{2}\sum_{e\in E} r(e)\theta(e)^2$$

Moreover, let *i* denote the electrical current flow obtained when the potential is constant on both A and Z, normalized so that the net flow from A to Z is 1. Thomson's Principle states that if θ is any unit flow from A to Z and *i* is the unit current flow from A to Z, then:

$$\varepsilon(\theta) \geq \varepsilon(i)$$

With equality iff $\theta = i$.

The proof of Thomson's Principle is completed by using orthogonal decomposition. This process starts in a Hilbert space $l^2\ (E,r)$ of antisymmetric edge functions with inner product:

$$(\phi,\psi)_r = \frac{1}{2}\sum_{e\in E} r(e)\phi(e)\psi(e)$$

Then, $\varepsilon(\theta) = \|\theta\|_r^2$.

To gain further understanding on Thomson's Principle, several essential concepts must be explained:

- Star space refers to the span of star vectors at vertices. The star at vertex x is the antisymmetric function that is $c(e)$ on each outgoing orientation and $-c(e)$ on the reverse.
- Cycle space refers to the span of antisymmetric indicator vectors of oriented cycles.
- Kirchhoff's Current Law depicts that the electric current i is orthogonal to every cycle. So, i is the orthogonal projection of any flow θ with the same divergence onto the star space (1).

To proceed, let's take any unit flow θ with the same divergence as i. Decomposing orthogonally gives:

$\theta = i + (\theta - i)$

With $i \in$ *star space* and $(\theta - i) \in$ *cycle space*. Orthogonality gives:

$$\mathcal{E}(\theta) = \|i\|_r^2 + \|\theta - i\|_r^2 \geq \|i\|_r^2 = \mathcal{E}(i)$$

and with equality iff $\theta - i = 0$, hence $\theta = i$.

Upon the completion of the proof, it is understood that Thomson's Principle identifies the electric current as the energy-minimizing unit flow. Since effective resistance corresponds to the energy with unit current, then Thomson's Principle reveals the energy characterization used frequently:

$R = min_{\theta\ unit}\ \varepsilon(\theta)$

As a result, the transience property of a random walk, in relation to electrical networks can be proven by constructing any unit flow to infinity with finite energy (1). This also sets the ground for understanding the behavior of resistances in modified networks as formalized by Rayleigh's Monotonicity Principle.

Rayleigh's Monotonicity Principle

Rayleigh's Monotonicity Principle depict that increasing conductances, which mirrors adding edges in graphs, either decreases effective resistance or it remains unchanged. Similarly, decreasing resistances cannot increase effective resistance.

To expand, let c and c' be two assignments of conductances on the same graph and $c(e) \leq c'(e)$ for every edge e. Let $C_c(A\leftrightarrow Z)$ denote effective conductance from A to Z and similarly for c' (6). Then:

$C_c(A\leftrightarrow Z) \leq C_{c'}(A\leftrightarrow Z)$

So:

$R_c(A\leftrightarrow Z) \geq R_{c'}(A\leftrightarrow Z)$

The proof of this principle can be derived by using current and energy:

Let i_c be the unit current for conductances c and $i_{c'}$ be the unit current for c'. Then, the energies of the conductances are $\varepsilon_c(i_c) = R_c(A\leftrightarrow Z)$ and $\varepsilon_{c'}(i_{c'}) = R_{c'}(A\leftrightarrow Z)$. If these two energies are compared by considering that $r_{c'} \leq r_c$, then:

$$\varepsilon_c(i_c) = \sum_e r_c(e)\, i_c(e)^2 \geq \sum_e r_{c'}(e)\, i_c(e)^2 = \varepsilon_{c'}(i_c)$$

Since Thomson's Principle states that the energy of any unit flow in the c' network is at least the minimal energy $\varepsilon_{c'}\ (i_{c'})$, then:

$\varepsilon_{c'}(i_c) \geq \varepsilon_{c'}(i_{c'})$

As a result:

$R_c(A\leftrightarrow Z) = \varepsilon_c(i_c) \geq \varepsilon_{c'}\ (i_{c'}) = R_{c'}\ (A\leftrightarrow Z)$

Since resistance is the reciprocal of conductance, then:

$C_c \leq C_{c'}$

In short, if edges are added to a graph, conductances are increased, then effective resistance to infinity can only decrease; so, the transience property of the random walk can only be preserved (1). On the other hand, if edges are deleted, resistance can only increase; this showcases that removing many edges could convert a transient graph into a recurrent one. However, bounded multiplicative changes in conductances do not change property of the random walk, resulting in either the transience or recurrence being preserved.

After the use of Rayleigh's Principle to showcase how resistance behaves under structural modification, the next section will utilize Nash–Williams Criterion to obtain a quantitative condition for determining recurrence.

Nash-Williams Criterion

The final principle crucial for proving Pólya's Theorem through electrical network theory is the Nash-William's Inequality and Criterion. This criterion depends on the possibility of finding infinitely many pairwise edge-disjoint cutsets separating the origin from infinity. If this possibility is fulfilled, then the conductances of these cutsets are small enough so that the sum over their reciprocal total conductances diverges. As a result of this divergence, the network that this criterion is applied to is deemed as recurrent

(1). The lower bound on effective resistance for this recurrence to exist is given by the Nash-Williams criterion.

The framework needed to understand the Nash-Williams Criterion starts by setting up a finite graph G that has positive conductances, so $c(e) > 0$. Then, a set of edges $\Pi \subset E$ is noted as a cutset that separates a, which is a fixed vertex, from infinity, if every path possible from the fixed vertex to infinity uses at least one edge of the cutset.

Let a and z be distinct vertices of a finite network and let Π_1, Π_2, Π_3 ... Π_n be pairwise disjoint cutsets that separate the above vertices. The Nash-William Inequality and Criterion are then expressed as follows:

$$R(a \leftrightarrow z) \geq \sum_{k=1}^{n} \Big(\sum_{e \in \Pi_k} c(e) \Big)^{-1}$$

Since,

$$C(\Pi_k) = \sum_{e \in \Pi_k} c(e)$$

Then,

$$R(a \leftrightarrow z) \geq \sum_{k=1}^{n} \frac{1}{C(\Pi_k)}$$

As a result, if$\{\Pi_n\}_{n\geq 1}$ are pairwise disjoint finite cutsets that separate a, the fixed vertex, from infinity (6), and:

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{e \in \Pi_k} c(e)} = \infty$$

Then,

$$R\ (a \leftrightarrow \infty) = \infty$$

So, the network is recurrent.

Furthermore, the proof of the Nash-Williams Inequality comes from the application of Cauchy-Schwarz Inequality. Let i be the unit current flow from a to z. For a single cutset Π separating a from z define the oriented boundary such that edges in Π are counted when oriented from side a to side z. Then the net current across Π equals 1. Moreover, let M be the component containing a when the edges of Π are removed and also let Z be the set of vertices on the other side of Π. (1) The current conservation shows:

$$\sum_{\substack{e \in \Pi \\ e \text{ is oriented from } K \text{ to } Z}} i(e) = 1$$

Applying Cauchy-Schwarz to the finite set Π reveals:

$$\Big(\sum_{e \in \Pi} |i(e)| \Big)^2 \leq \Big(\sum_{e \in \Pi} r(e) i(e)^2 \Big) \Big(\sum_{e \in \Pi} c(e) \Big)$$

However, since the left-hand side is a sub-sum, then:

$$\sum_{e \in \Pi} r(e) i(e)^2 \leq \sum_{e \in E} r(e) i(e)^2 = \mathcal{E}(i) = R(a \leftrightarrow z)$$

Since the left-hand side is also equal to minimum 1, it follows that:

$$1 \leq \Big(\sum_{e \in \Pi} |i(e)| \Big)^2 \leq R(a \leftrightarrow z) \Big(\sum_{e \in \Pi} c(e) \Big)$$

The rearrangement of the inequality above gives:

$$R(a \leftrightarrow z) \geq \frac{1}{\sum_{e \in \Pi} c(e)}$$

If this inequality is applied to each of Π_k, and then they are added together and the final inequality proves the Nash-Williams Criterion:

$$R(a \leftrightarrow z) \geq \sum_{k=1}^{n} \frac{1}{\sum_{e \in \Pi_k} c(e)}$$

The proof depicted above applied to finite networks; the proof for infinite networks is done by exhausting the graph, which allows the result to be achieved by using conventional arguments. Having established a sound analytical base for examining random walks through network theory by the use of these three governing principles, the next section applies them systematically to prove Pólya's Theorem.

Proving Pólya's Theorem

This section serves as the core for the study as it merges with the knowledge presented above to explore concepts that depict the relationship between electrical networks and probability theory. Pólya's Theorem, in its most basic explanation, showcases that random walks are recurrent in the first and second dimension, and then they become transient in the third dimension (2).

<u>Transience and Recurrence</u>

First of all, the connection between transience and recurrence and electrical networks needs to be firmly established. If recurrence is present in an electrical network, then any unit flow to infinity has infinite energy in this network. Due to the relationship between energy and effective resistance, if there is infinite energy then the effective resistance to infinity is infinite. This is due to the fact that the current must flow through infinitely many revisits, which costs infinite energy in this network (1). On the other hand, transience occurs when there exists a unit flow to infinity with finite energy. Since transience means the walk has a positive chance of escaping forever, in a transient network, current can flow to infinity with finite energy, so the effective resistance is finite. In short:

- If the walk is recurrent: $R_{eff}(0 \leftrightarrow \infty) = \infty$
- If the walk is transient: $R_{eff}(0 \leftrightarrow \infty) < \infty$

The First Dimension: $\mathbb{Z}^d$, $d = 1$

In the first dimension, the graph consists of integers and each edge has a resistance of 1. These edges are also between x and $x \pm 1$. If the resistance from zero to infinity is taken into consideration, then there is exactly one simple chain of edges in each direction as with each step only plus or minus 1 is moved. Because of this, the path in the first dimension consists of a series chain of resistance. As a result, effective resistance is:

$$R_{eff}(0 \leftrightarrow \infty) = \sum_{x=0}^{\infty} r(\{x, x+1\}) = \sum_{k=0}^{\infty} 1 = \infty$$

Since there are infinitely many edges with resistance 1 and $R_{eff}(0 \leftrightarrow \infty) = \infty$, then the walk is recurrent for $\mathbb{Z}^d$, $d = 1$ as it diverges (9).

The Second Dimension: $\mathbb{Z}^d$, $d = 2$

For the second dimension, the edge resistance is 1 for each edge as in the previous case. Establishing recurrence in this dimension requires constructing a sequence of disjoint cutsets, each of which separates 0 from infinity. If the origin is fixed at 0, then every n-th shell of where $n \geq 1$ can be defined as:

$$S_n = \{x \in \mathbb{Z}^2 : \|x\|_1 = n\}$$

Furthermore, the set of edges must be denoted as Π_n and one end of it must be at $\{x : \|x\|_1 \leq n\}$ and the other at $\{x : \|x\|_1 \geq n + 1\}$. Each Π_n is finite, separate zero from infinity and pairwise disjoint. In this context, S_n can be deemed as a square centered at 0 and its side length is $2n + 1$. Also, C_n is the set of edges that crosses C_n to C_{n+1}. Its size satisfies:

$$|C_n| \sim 8n$$

as the perimeter of the square with side lengths $2n + 1$ is approximately $8n$.

The Nash-Williams Inequality states that for any finite set of pairwise disjoint cutsets $\Pi_1, \Pi_2, \Pi_3 \ldots \Pi_k$:

$$R_{eff}(0 \leftrightarrow z) \geq \sum_{n=1}^{k} \frac{1}{\sum_{e \in \Pi_n} c(e)}$$

Since $c(e) = 1$ due to unit conductances, then:

$$R_{eff}(0 \leftrightarrow z) \geq \sum_{n=1}^{k} \frac{1}{|\Pi_n|}$$

If this inequality is applied to an infinite sequence of cutsets $\Pi_1, \Pi_2, \Pi_3 \ldots$:

$$R_{eff}(0 \leftrightarrow \infty) \geq \sum_{n=1}^{k} \frac{1}{|\Pi_n|} = \sum_{n=1}^{k} \frac{1}{8n} = \frac{1}{8} \sum_{n=1}^{k} \frac{1}{n} = \infty$$

Since:

$$R_{eff}(0 \leftrightarrow \infty) \geq \sum_{n=1}^{\infty} \left(\sum_{e \in C_n} c(e) \right)^{-1} \geq \sum_{n=1}^{\infty} \frac{1}{|C_n|} \approx \sum_{n=1}^{\infty} \frac{1}{n}$$

As the obtained inequality is a harmonic series, it diverges. So:

$$R_{eff}(0 \leftrightarrow \infty) = \infty$$

As a result, the walk is recurrent for $\mathbb{Z}^d$, $d = 2$.

The Third Dimension: $\mathbb{Z}^d$, $d = 3$

For the third dimension, in order to prove transience, a unit flow θ from the origin to infinity is defined which has finite energy (1). In order to achieve this outcome, a random ray that undergoes radial flow is constructed.

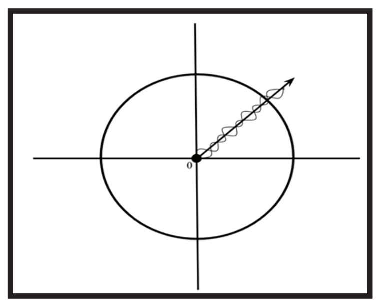

Figure 3. The Diagram of the Constructed Ray (1) The diagram shows a random ray beginning at the origin and extending radially to infinity, representing unit flow. If this ray is shown to have finite energy, it confirms the onset of transience for random walks in dimensions d≥3.

If a random direction, denoted with D, is chosen from the origin in dimension $\mathbb{R}^3$; then each ray of D can be closely approximated by a simple lattice path $P(D)$ from zero to infinity. For each path P there is a unit flow ϑ_P that is capable of sending +1 along each oriented edge traversed by P. Since each $P(D)$ is a unit path from zero to infinity, there must be a unit flow θ from zero to infinity:

$$\theta(e) := P_D(forwards) - P_D(backwards)$$

To continue, an estimate of energy must be made where $r(e) = 1$:

$$\mathcal{E}(\theta) = \frac{1}{2}\sum_{e \in E} r(e)\theta(e)^2$$

$\theta(e)$ must be bounded:

$$|\theta(e)| \leq P_D[e \in P_D]$$

So:

$$\theta(e)^2 \leq P_D[e \in P_D]^2$$

Furthermore, if an edge e is fixed and if it has Euclidean distance approximately R from the origin, then geometric reasoning can be used to give the bound (1):

$$P_D[e \in P_D] \leq \frac{A}{R^2}$$

for a constant A. Also, there are at most BR^2 edges whose centers lie at Euclidian radius R and $R + 1$ for a constant B. To conclude, contribution to the energy from R can be, at maximum:

$$\sum \theta(e)^2 \leq (BR^2) \times \left(\frac{A}{R^2}\right)^2 = \frac{BA^2}{R^2}$$

If summed over $R \geq 1$; then:

$$\mathcal{E}(\theta) \leq \sum_{R=1}^{\infty} \frac{BA^2}{R^2}$$

Since $\sum \frac{1}{R^2}$ converges:

$$\mathcal{E}(\theta) \leq \sum_{R=1}^{\infty} \frac{BA^2}{R^2} < \infty$$

Thus, θ is a unit flow from zero to infinity that has finite energy. By Thomson's Principle:

$$R_{eff}(0 \leftrightarrow \infty) = \mathcal{E}(\theta) < \infty$$

So, the walk is transient for $\mathbb{Z}^d$, $d = 3$.

Having proved Pólya's Theorem and demonstrated the dimensional structure of recurrence and transience, the analysis now extends to explore how these properties respond to modifications of the underlying graph.

Graph Modifications

This section revolves around delineating how modifications to graphs affect effective resistance and thus the transience or recurrence probabilities. Two different cases will be explored in this section: how the properties of random walks are affected when finite edges are added or deleted; and how addition or deletion of infinite edges changes transience and recurrence.

<u>Finite Modifications</u>

In a graph that has been arranged according to electrical network theory, understanding the changes in transience and recurrence for the addition or deletion of finite edges comes from Rayleigh's Monotonicity Principle. This principle, explored in detail in the previous sections, showcases that if edges are added to a graph, the conductances increase, which can only reduce the effective resistance to infinity; therefore, a transient random walk remains transient. Conversely, deleting edges increases the resistance, meaning that removing enough edges could transform a transient graph into a recurrent one. However, if conductances are altered only by bounded multiplicative factors, the nature of the random walk remains unchanged.

<u>Infinite Modifications</u>

After understanding the effects of finite modifications, the next question that arises relates to how addition or deletion of infinite edges can change recurrence or transience. However, one must first understand whether adding or deleting infinite edges can alter any property in the first place. This exploration starts with a graph $G = (V, E)$ which represents a locally finite, connected graph, where V is the set of all vertices in the graph and E is the set of all edges. Furthermore, the conductances in this graph are unit conductances. The effective resistance to infinity from vertex o is:

$$R_{eff}(o \leftrightarrow \infty) = \inf_{\theta \in \mathcal{F}_{a \to b}} \mathcal{E}(\theta)$$

Furthermore, if $\{\Pi_n\}_{n \geq 1}$ are pairwise disjoint finite cutsets that separate the fixed vertex from infinity, then:

$$R_{eff}(o \leftrightarrow \infty) \geq \sum_{k=1}^{n} \frac{1}{\sum_{e \in \Pi_k} c(e)}$$

If a sub-graph $G' = (V, E \setminus F)$ is formed by deleting the edges of $F \subseteq E$; Rayleigh's Monotonicity Principle states that:

$$R_{eff,G'}(o \leftrightarrow \infty) \geq R_{eff,G}(o \leftrightarrow \infty),$$

with strict inequality if the deletion increases the length of some path from o to infinity. Whilst this relation concerns edge deletion, if infinitely many edges are removed while the graph remains connected, and if a sequence of cutsets $\{\Pi'_n\}$ in G' satisfies:

$$\sum_{k=1}^{n} \frac{1}{\sum_{e\in\Pi'_k} c(e)} = \infty$$

then by the Nash-Williams Criterion:

$$R_{eff,G'}(o \leftrightarrow \infty) = \infty$$

So, the walk is recurrent. On the other hand, if a sub-graph $G'' = (V, E \cup F)$ is formed by adding the edges of F; Rayleigh's Monotonicity Principle states that:

$$R_{eff,G''}(o \leftrightarrow \infty) \leq R_{eff,G}(o \leftrightarrow \infty)$$

with strict inequality if the added edges introduce shorter paths to infinity or increase the total conductance of cutsets. In case of infinite additions, if infinitely many edges are added such that, for some sequence of cutsets $\{\Pi_k\}$(1) :

$$\sum_{k=1}^{n} \frac{1}{\sum_{e\in\Pi_k} c(e)} < \infty$$

So:

$$R_{eff,G''}(o \leftrightarrow \infty) < 0$$

Hence the network becomes transient.

In summary, while finite modifications preserve recurrence and transience, infinite modifications may alter these properties depending on whether the sum diverges or converges after modification.

The last concept that needs to be evaluated is the k-fuzz. A k-fuzz occurs when edges are added not only between nearest neighbors but also between vertices that can be traversed within "k" steps. Since this study is concerned with the relationship between electrical networks and random walks, the occurrence of a k-fuzz reflects inserting new parallel pathways that reduce the effective resistance between vertices. In the preceding sections, through Rayleigh's Monotonicity Principle, it was shown that the effective resistance to infinity can only decrease under such additions. On the other hand, k-fuzz encapsulates certain differences; mainly, it does not alter the fundamental recurrence/transience classification. As a result, it can be understood that recurrence and transience are robust under bounded local modifications of the graph which is evidenced by the concept of the k-fuzz (1). Building on all preceding analyses, the next section summarizes the overall results derived throughout the study.

RESULTS

This exploration was structured around three concepts: random walks, electrical networks, and Pólya's Theorem. By integrating these concepts, the study sought to establish a rigorous understanding of the relationship between probability theory and electrical network theory. The investigation began with a detailed rederivation of Pólya's theorem using the electrical network framework. By explicitly computing the effective resistance to infinity:

$$R_{eff}(o \leftrightarrow \infty) = \inf_{\theta\in\mathcal{F}_{a\to b}} \mathcal{E}(\theta)$$

the investigation confirmed that effective resistance is infinite for the first and second dimensions and the effective resistance is smaller than infinity for dimensions bigger or equal to three. Also, these calculations were supported by the Nash–Williams criterion, which primarily used cutsets for the proofs. The second stage of the analysis examined graph modifications. Applying Rayleigh's Monotonicity Principle, it was demonstrated that edge deletion satisfies:

$$R_{eff,G'}(o \leftrightarrow \infty) \geq R_{eff,G}(o \leftrightarrow \infty)$$

and edge addition satisfies:

$$R_{eff,G''}(o \leftrightarrow \infty) \leq R_{eff,G}(o \leftrightarrow \infty)$$

In terms of contributions, the study advances two main areas. First, it confirms the findings of Pólya's Theorem, a central theorem in probability theory that has been extensively examined through mathematical proofs. By employing electrical network theory to verify this theorem, the study provides a more rigorous and interdisciplinary confirmation. While previous research has primarily utilized purely mathematical methods, there has been limited research using physical analogies. This paper addresses that gap by showcasing how principles from physics can be used to reaffirm a probabilistic result.

Secondly, the paper contributes to understanding the stability of random walks under structural modifications. It clearly proves that extensive infinite deletions can induce recurrence, while large infinite additions can lead to transience. Conversely, random walks remain stable under finite modifications, as adding or removing finitely many edges does not qualitatively change the effective resistance. However, this stability breaks under infinite modifications, which can result in altering the recurrence or transience classification entirely.

In short, this exploration bridges classical probabilistic theory and electrical network analysis. It demonstrates that recurrence corresponds to infinite effective resistance, while transience corresponds to finite effective resistance. Moreover, the study quantifies how these properties respond to both finite and infinite structural perturbations.

DISCUSSION

The findings of this study disclose a deeper theoretical coherency between random walks and electrical networks, highlighting the common structural background shared by stochastic and physical systems. By grounding Pólya's theorem within the framework of effective resistance, the research reinterprets recurrence and transience not as isolated probabilistic behaviors but as manifestations of the broader energetic and geometric characteristics of infinite networks. This synthesized viewpoint positions the electrical network formulation as a unifying analytical principle bridging probability theory, physics, and graph theory.

A central insight emerging from this analysis is how structural perturbations affect the stochastic stability The preservation of recurrence and transience under finite modifications reveals their nature as global topological invariants rather than local irregularities. In contrast, the breakdown of stability under large-scale or infinite modifications emphasizes the delicate balance between local and global connectivity in determining long-term stochastic behavior. This relation between behavior and structure invites further research of how topological invariants might predict critical transitions between recurrent and transient regimes.

The analysis also reframes the dimensional threshold at two dimensions as an energetic boundary rather than a solely geometric one. From the continuum analogy, the effective resistance to infinity in a d-dimensional resistive medium satisfies (8):

$$R_{eff}(o \leftrightarrow \infty) \approx \int_a^{\infty} \frac{dr}{r^{d-1}}$$

The divergence of the integral in lower dimensions reflects the incapability of the system to dissipate potential efficiently, whereas its convergence in higher dimensions signifies sufficient spatial capacity for energy dispersion. This physical analogy deepens mathematical intuition, demonstrating how dimensionality serves as a critical boundary between confinement and dispersion of stochastic energy.

More broadly, this interdisciplinary approach underscores the reciprocal validation between mathematics and physics. By employing Thomson's and Rayleigh's laws to verify mathematical theorems, the study exemplifies how cross-disciplinary reasoning can enhance the rigor and applicability of abstract results. Beyond extending and reinforcing established results, this approach also offers a framework for integrating physical insight into the construction of proofs in complex stochastic systems.

Within the context of existing literature, this paper extends prior work, especially those by Lyons and Peres (1) and by Doyle and Snell (6), and advances the field by providing physically motivated and analytically rigorous proofs in all three dimensions. Whereas earlier studies primarily emphasized probabilistic proofs, the present research establishes a physically grounded derivation unifying energetic and geometric perspectives. In doing so, the study not only confirms and extends Pólya's Theorem through a novel interdisciplinary lens, but also it inaugurates the electrical network analogy as a robust framework for future studies exploring the stability and universality of stochastic processes in mathematical physics.

CONCLUSION

This study contributes a new theoretical perspective on random walks by showing how the principles of electrical network theory can rigorously characterize recurrence and transience. Its novelty lies not in confirming Pólya's Theorem, but in reframing it through the merging of different disciplines. By setting this interdisciplinary connection, the research transforms an established probabilistic theorem into a physically interpretable framework with analytical depth and conceptual coherence, thereby filling a critical gap in the existing research regarding the interplay between physics and mathematics.

The strongest outcome of this work is the articulation of structural stability criteria for random walks: finite modifications preserve qualitative behavior, while infinite perturbations may alter it. This distinction between local robustness and global sensitivity opens a pathway toward a broader theory of stochastic stability in networked systems. The study thus not only verifies known results through a new method but also extends the scope of inquiry by exposing how topology and energy interact to shape probabilistic dynamics.

Nevertheless, the research remains bounded by its reliance on idealized, symmetric lattices. The proofs assume invariance, symmetry, and regular connectivity, however, these conditions may not hold in irregular or weighted networks. Additionally, while infinite modifications were analytically framed, their full dynamical implications were not exhaustively explored.

Future research should extend this inter-disciplinary reasoning beyond Euclidean lattices to non-Euclidean and disordered graphs, where geometric irregularity and degree heterogeneity may yield different thresholds for random walk characteristics. Research with different graph types may enhance the relation between stochastic behaviour and physical networks. Furthermore, investigating marginal networks, those with logarithmically diverging effective resistance, could provide further implications for the transitions between stable and unstable regimes in random walks. In a broader sense, the integration of this dual approach –physical and mathematical- with computational and probabilistic modeling would further examine its robustness and reveal new connections between structure, flow, and randomness.

To conclude, this research reformulates the study of random walks as an interdisciplinary inquiry where energy, geometry, and probability merge within infinite systems, laying conceptual groundwork for future advances at the intersection of mathematics and physics.

ACKNOWLEDGMENTS

Thanks to Professor Sourav Sarkar for encouragement, support and mentorship throughout the research and contribution to the author's understanding of probability theory, random walks and electrical networks.

CONFLICT OF INTEREST

The author declares that there are no conflicts of interest regarding the publication of this article.

REFERENCES

1. Lyons R, and Peres Y. Probability on Trees and Networks. *Cambridge University Press*. 2016. ISBN: 978-1-108-73272-7, 18-48.
2. Pólya G. Collected Papers, The MIT Press, Cambridge, Massachusetts, (IV): p. 582–585.
3. Hoel PG, Port SC, and Stone CJ. Introduction to Probability Theory. Waveland Press. 1971. ISBN: 0-395-04636-x, 216-230.
4. Krishnapur M, and Peres Y. Recurrent Graphs Where Two Independent Random Walks Collide Finitely Often. U.C. Berkeley Department of Statistics. 2004; p.1-10. https://doi.org/10.1214/ECP.v9-1111
5. Johnston D. An Introduction to Random Walks. Self-published manuscript. 2011; p.1-11.
6. Doyle PG & Snell JL. Random Walks and Electric Networks. *Mathematical Association of America*. 2000; p.1-116.
7. Harry F. *The importance of Kirchhoff's Laws in electrical engineering*. Student Circuit. 2024, April 29. Retrieved from https://www.student-circuit.com/learning/the-importance-of-kirchhoffs-laws-in-electrical-engineering/ (accessed October 18, 2025)
8. What is energy? Explained- U.S. Energy Information Administration. (n.d.). EIA. Available from: https://www.eia.gov/energyexplained/what-is-energy/ (accessed November 8, 2025)
9. Mare S. Pólya's Recurrence Theorem. Dartmouth Mathematics. 2013; p.1-5.